9. VAKULENKO A.A., On brittle fracture under creep, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 6, 1982.
10. BELLMAN R. and COOK K.L., Differential-Difference Equations, Mir, Moscow, 1967.
11. CHEN I.-W. and ARGON A.S., Creep cavitation in 304 stainless steel, Acta Met., 29, 7, 1981.

Translated by M.D.F.

PMM U.S.S.R., Vol.53, No.5, pp.665-668, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

COMPLETE CONTROLLABILITY OF LINEAR DYNAMIC SYSTEMS*

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A complete controllability criterion is suggested for a linear dynamic system with bounded controls. It is shown that programmed control, taking the system from one state to another, can be constructed in quasipolynomial form. The problem of constructing such a control thus basically reduces to solving a linear system of equations.

1. One of the fundamental results of control theory is the Kalman criterion /1/, which provides the necessary and sufficient conditions of complete controllability of dynamic systems of the form

$$x^{\cdot} = Ax + Bu, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m \tag{1.1}$$

Here, $A.\mathbf{R}^n \to \mathbf{R}^n, B: \mathbf{R}^m \to \mathbf{R}^n$ are time-independent linear operators. The Kalman criterion states that the pair of matrices A, B should satisfy the following condition of general position:

$$\operatorname{rank}(B, \ AB, \ \ldots, \ A^{n-1}B) = n \tag{1.2}$$

This condition ensures that any point $x_0 \in \mathbf{R}^n$ is reachable from any point $x_1 \in \mathbf{R}^n$ by moving along a trajectory of the dynamic system (1.1) with some control $u = u(t)$.

In this paper, we present an analogue of the Kalman criterion for the case when the controls u in Eq.(1.1) are bounded,

$$|u| \leqslant C \tag{1.3}$$

and we also provide a technique for constructing a programmed control that achieves a transition between states. Given the constraint (1.3), the Kalman condition (1.2) is of course totally insufficient for complete controllability. Indeed, if all the eigenvalues of the matrix A have strictly negative real parts, then starting from any point $x_0 \in \mathbf{R}^n$ and moving along the trajectories of the system (1.1), (1.3), we will never be able to leave a certain bounded set, regardless of the choice of the matrix B. If conversely all the eigenvalues of the matrix A have strictly positive real parts, then, say, $0 \in \mathbf{R}^n$ is unreachable from a sufficiently distant point $x_0 \in \mathbf{R}^n$.

2. Let us discuss the following theorem, which was first proved in /2/ (for some further results, see /3, 4/).

Theorem 1. For complete controllability of system (1.1), (1.3), it is necessary and sufficient that, in addition to the Kalman condition (1.2), we also have

$$\operatorname{Re} \lambda_i = 0 \tag{2.1}$$

where λ_i are the eigenvalues of the matrix A.

Let us explain the need for condition (2.1). To fix our ideas, assume that the matrix A has the eigenvalue λ and

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Prikl. Matem. Mekhan., 53, 5, 845-848, 1989

$$\operatorname{Re}\lambda = a < 0 \tag{2.2}$$

Note that by replacing A by $-A$ we do not change the complete controllability properties of system (1.1), (1.3), and the "less than" sign in condition (2.2) does not restrict the generality of our analysis. Let D_T be the region of reachability in time T for system (1.1), (1.3) with initial condition $x(0) = 0$, and h_T the support function of the system. We have /5/

$$h_T(\xi) = \int \sup_{|u|\leqslant C} (\exp(A(T-t))Bu, \xi)\,dt \leqslant C\,|B|\int |\exp(A^*(T-t))\xi|\,dt \tag{2.3}$$

Here and in what follows, integration over t is from 0 to T, A^* is the transpose of the matrix A, and $|B|$ is the norm of the matrix B. From condition (2.2) it follows that there exists a vector $0 \neq \xi \in \mathbf{R}^n$ such that

$$|\exp(A^*t)\xi| \leqslant C\exp(at), \quad a < 0 \tag{2.4}$$

To this end, it suffices to take $\xi = x + \bar{x}$, where $x \in \mathbf{C}^n$ is the eigenvector of the matrix A^* with eigenvalue λ, which is not purely imaginary, and $\bar{x}$ is the complex-conjugate vector. From (2.3), (2.4) it follows that $h_T(\xi)$ is uniformly bounded in T, which contradicts complete controllability. Condition (2.1) is thus necessary.

3. We will now show that the control $u(t)$ taking the system from one point to another may be expressed as a vector quasipolynomial

$$u(t) = \sum a_{kl}\exp(\lambda_k t)\quad t^l, \quad a_{kl} \in \mathbf{C}^n \tag{3.1}$$

where λ_k are the eigenvalues of the matrix A or $-A$.

Theorem 2. Let system (1.1), (1.3) be completely controllable (which by Theorem 1 implies that conditions (1.2) and (2.1) are satisfied). Then the system may be taken from one given state to another by a control of the form (3.1).

Let the control $u(t)$ be such that the trajectory of system (1.1) passes through the points $x_0, x_1 \in \mathbf{R}^n$, $x(0) = x_0$, $x(T) = x_1$, $T \geqslant 0$. This is equivalent to

$$\int \exp(A(T-t))Bu(t)\,dt = x_1 - \exp(AT)x_0$$

We seek u in the form $u = u_0 + u_1$, where

$$\int \exp(A_i(T-t))Bu_i(t)\,dt = x_i(-1)^i, \quad A_i = A(-1)^i, \quad i = 0, 1 \tag{3.2}$$

If we find a quasipolynomial solution $u_i(t)$ of Eqs.(3.2) such that the norm $|u_i(t)|$ is small, e.g., does not exceed $C/2$, where C is the constant from (1.3), this will give a quasipolynomial control that takes the system (1.1), (1.3) from x_0 to x_1 in time T. Note that if the matrix A satisfies conditions (1.2), (2.1), then A_i also satisfies these conditions. Therefore, finally the problem reduces to solving the equation

$$\int_0^T \exp(A(T-t))Bu(t)\,dt = x \tag{3.3}$$

for some unknown polynomial $u(t)$ of small norm, with matrices A, B satisfying conditions (1.2), (2.1) and T a sufficiently long time.

We seek u in the form

$$u = u_{T,\xi}(t) = B^*\exp(A^*(T-t))\xi \tag{3.4}$$

Clearly (see /6/) u is a quasipolynomial. From (3.4) we naturally obtain the Euclidean norm $|\xi| = (\xi, \xi)^{1/2}$ of the vector ξ and also two additional norms and the operator P:

$$\|\xi\|_{\infty,T} = \sup_{0\leqslant t\leqslant T}|B^*\exp(A^*(T-t))\xi| = \sup_{0\leqslant t\leqslant T}|u_{T,\xi}(t)|$$
$$\|\xi\|_{2,T} = \left(\int |B^*\exp(A^*(T-t))\xi|^2\,dt\right)^{1/2} = \left(\int |u_{T,\xi}(t)|^2\,dt\right)^{1/2}$$
$$P\xi = P_T\xi = \int \exp(A(T-t))Bu_{T,\xi}(t)\,dt$$

In this notation, problem (3.3) reduces to solving the equation

$$P_T\xi = x \tag{3.5}$$

for small $(\leqslant C/2)$ norm $\|\xi\|_{\infty,T}$.

Note that the existence of the solution ξ of Eq.(3.5) (without a bound on the norm $\|\xi\|_{\infty,T}$) follows from the Kalman condition (1.2). We see that

$$(P_T\xi,\ \xi) = \|\xi\|^2_{2,T} \tag{3.6}$$

and (by the Kalman condition (1.2))

$$|\xi| \leqslant c\|\xi\|_{\infty,T} \tag{3.7}$$

where the constant c is independent of T.

The main fact that makes it possible to find a solution ξ of Eq.(3.5) with a small norm $\|\xi\|_{\infty,T}$ is supplied by the following lemma.

Lemma. Let condition (2.1) hold. Then as $T\to\infty$ we have $\|\xi\|^2_{2,T}/\|\xi\|^2_{\infty,T} \geqslant cT$, uniformly in $\xi\neq 0$, where c is a positive constant (independent of T and ξ).

Assuming that the lemma holds, let us estimate the norm $\|\xi\|_{\infty,T}$ of the solution ξ of Eq. (3.5). From (3.5), (3.6) and the lemma, we have

$$(x,\ \xi) = \|\xi\|^2_{2,T} \geqslant cT\|\xi\|^2_{\infty,T} \tag{3.8}$$

for large T. The Cauchy inequality and inequality (3.7) show that $c|x|\,\|\xi\|_{\infty,T} \geqslant |(x,\ \xi)|$. Now from (3.8)

$$\|\xi\|_{\infty,T} \leqslant c|x|/T \tag{3.9}$$

and therefore in particular for large T any solution ξ of Eq.(3.5) has a small norm $\|\xi\|_{\infty,T}$. From inequality (3.9) it also follows that we can go from the point $0\in\mathbf{R}^n$ to the point $x\in\mathbf{R}^n$ or in the opposite direction in time $T = 0(|x|)$

It remains to prove the lemma. Let $u(t) = u_{T,\xi}(t) = \Sigma a_{kl}\exp(\lambda_k t)\,t^l,\ a_{kl}\in\mathbf{C}^n$. Then $\operatorname{Re}\lambda_k = 0$ by condition (2.1).

We have $u(t) = \Sigma p_k(t)\exp(i\omega_k t)$, where p_k are (vector-valued) polynomials whose degree does not exceed the maximum dimension of the Jordon cells of the matrix A with eigenvalue $i\omega_k$, where ω_k is real. We have to show that the function u of this form satisfies, as $T\to\infty$, the inequality

$$I = \int |u(t)|^2\,dt \geqslant cT\|u\|^2_{\infty,T},\quad \|u\|_{\infty,T} = \sup_{0\leqslant t\leqslant T}|u(t)| \tag{3.10}$$

Indeed,

$$I = T\langle|u(T\tau)|^2\rangle = T\left\langle\left|\sum p_k(T\tau)\exp(i\omega_k T\tau)\right|^2\right\rangle = T\sum J_k + T\sum_{k\neq l}K_{kl} \tag{3.11}$$

$$J_k = \langle|p_k(T\tau)|^2\rangle,\quad K_{kl} = \langle(p_k(T\tau),\,p_l(T\tau))\exp i(\omega_k-\omega_l)T\tau\rangle$$

$$\langle f(\tau)\rangle = \int_0^1 f(\tau)\,d\tau,\quad (x,y) = \sum x_i\bar{y}_i,\ x,y\in\mathbf{C}^n$$

($(,)$ is the standard Hermitian scalar product).

Let $p_{k,T}(t) = p_k(Tt)$. Then clearly

$$J_k \geqslant c_1\|p_{k,T}\|^2_{\infty,1} = c_1\|p_k\|^2_{\infty,T} \tag{3.12}$$

where the constant c_1 depends only on the degree of the polynomials p_k, or equivalently on the dimension of the Jordan cells of the matrix A. More-accurate calculations using Legendre polynomials show that c_1 may be taken in the form $(1/\deg p_k)^2$. The second term in (3.11) can be estimated using integration by parts:

$$|TK_{k,l}| = \left|-\left\langle\frac{\exp(i(\omega_k-\omega_l)T\tau)}{i(\omega_k-\omega_l)}\frac{d}{d\tau}(p_{k,T},p_{l,T})(\tau)\right\rangle + \frac{\exp(i(\omega_k-\omega_l)Tt)}{i(\omega_k-\omega_l)}(p_{k,T},p_{l,T})(t)\Big|_0^1\right| \leqslant \Omega^{-1}\left(\left\langle\left|\frac{dp(\tau)}{d\tau}\right|\right\rangle + 2\|p\|_{\infty,1}\right), \tag{3.13}$$

$$p = (p_{k,T},\,p_{l,T}),\quad \Omega = \min_{k\neq l}(\omega_k-\omega_l)$$

We have the obvious bound

$$\langle|dp(\tau)/d\tau|\rangle \leqslant c_2\|p\|_{\infty,1} \tag{3.14}$$

with some constant c_2, and the known results of approximation theory (see /7/) suggest that $c_2\leqslant(\deg p)^2$ and even a tighter bound $c_2\leqslant\langle|dT_N(\tau)/d\tau|\rangle$, where $N=\deg p$, T_N is a Chebyshev polynomial.

Collecting inequalities (3.12)-(3.14), we obtain for the second term in (3.11) the majorant

$$\Omega^{-1}c_2 \sum_{k \neq l} \| p_k \|_{\infty, T} \| p_l \|_{\infty, T} \leqslant \Omega^{-1}c_2 \left(\sum \| p_k \|_{\infty, T}\right)^2 \qquad (3.15)$$

where the constant c_2 depends only on the dimensions of the Jordan cells of the matrix A (or, equivalently, on the degrees of the polynomials p_k) and can be bounded by

$$c_2 \leqslant \max_{k \neq l} (\deg p_k + \deg p_l)^2 + 2$$

From inequalities (3.11), (3.12), and (3.15), we obtain

$$I \geqslant c_1 T \sum_{k=1}^{M} \| p_k \|^2_{\infty, T} - c_2 \Omega^{-1} \left(\sum_{k=1}^{M} \| p_k \|_{\infty, T} \right)^2 \geqslant$$
$$\geqslant (c_1 T/M - c_2/\Omega) \left(\sum \| p_k \|_{\infty, T}\right)^2 \geqslant (c_1 T/M - c_2/\Omega) \| u \|^2_{\infty, T}$$

which proves (as $T \to \infty$) inequality (3.10) and the lemma. This completes the proof of Theorem 2.

4. As an example, consider a mechanical system consisting of N pendulums with a common suspension point. This suspension point moves in a controlled motion with bounded acceleration. In the linear approximation, the equations of motion have the form

$$x^{\cdot\cdot} = u, \quad x_i^{\cdot\cdot} + \omega_i^2 x_i = u, \quad i = 1, \ldots, N, \quad | u | \leqslant 1$$

Here x is the displacement of the suspension point, and x_i is the displacement of the i-i-th pendulum. The control problem is to bring the system to rest, i.e., to take it from a given initial state to a state in which the displacements and the velocities are zero for all the pendulums and the suspension point. The previous results suggest that the problem is solvable for any initial state if and only if the frequencies $| \omega_i | \neq 0$ are all different (this is an interpretation of the Kalman condition (1.2)). Following the proof of Theorem 2, we can also obtain a bound on the relaxation time T under quasipolynomial control u of the form

$$u(t) = \xi_1 + \xi_2 t + \operatorname{Re} \sum a_k \exp(i\omega_k t), \qquad a_k \in \mathbf{C}$$

The final result is expressed by the inequality ($\mathbf{x}$ is the initial state vector)

$$T \leqslant 2\sqrt{N+2}\, | \mathbf{x} | + 4((N-1)/\Omega + \sqrt{14N}/\omega) \qquad (4.1)$$
$$\mathbf{x} = (x, x^{\cdot}, x_1, x_1^{\cdot}, \ldots, x_N, x_N^{\cdot})$$
$$| \mathbf{x} |^2 = x^2 + (x^{\cdot})^2 + \sum (\omega_i x_i)^2 + (x_i^{\cdot})^2$$
$$\Omega = \min_{i, j=1, \ldots, N} | \omega_i \pm \omega_j |, \qquad \omega = \min_{i=1, \ldots, N} | \omega_i |$$

A detailed derivation of the bound (4.1) is similar to the proof of Theorem 2 and requires fairly lengthy calculations.

Similar results were obtained by Chernous'ko in the problem of the relaxation of a system of N pendulums (without relaxation of the suspension point), and in the problem of the relaxation of a single pendulum with its suspension point.

I would like to acknowledge the useful comments by F.L. Chernous'ko. In particular, he suggested the programmed control formula (3.4) and used it to derive an explicit bound on the relaxation time for a system of pendulums with a common suspension point.

REFERENCES

1. KRASOVSKII N.N., Theory of Motion Control, Nauka, Moscow, 1968.
2. BRAMMER R.F., Controllability of linear autonomous systems with positive passive controllers, SIAM J. Contr., 10, 2, 1972.
3. KOROBOV V.I., MARINICH A.P. and PODOL'SKII E.N., Controllability of linear autonomous systems with control constraints, Diff. Uravn., 11, 11, 1975.
4. FORMAL'SKII A.M., Controllability and Stability of Systems with Resource Constraints, Nauka, Moscow, 1974.
5. IOFFE A.D. and TIKHOMIROV V.M., Theory of Extremal Problems, Nauka, Moscow, 1974.
6. ARNOL'D V.I., Ordinary Differential Equations, Nauka, Moscow, 1984.
7. TIKHOMIROV V.M., Approximation theory, Itogi Nauki i Tekhniki. Sovremennye problemy Matematiki. Fundamental'nye Napravleniya, Nauka, Moscow, 14, 1987.

Translated by Z.L.